



A note on the cylindrical solitary waves in an electron-acoustic plasma with vortex electron distribution

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In the present work, we consider the propagation of nonlinear electron-acoustic non-planar waves in a plasma composed of a cold electron fluid, hot electrons obeying a trapped/vortex-like distribution, and stationary ions. The basic nonlinear equations of the above described plasma are re-examined in the cylindrical coordinates through the use reductive perturbation method in the long-wave approximation. The modified cylindrical Korteweg–de Vries equation with fractional power nonlinearity is obtained as the evolution equation. Due to the nature of nonlinearity, which is fractional, this evolution equation cannot be reduced to the conventional Korteweg–de Vries equation. An analytical solution to the evolution equation, by use of the method developed by Demiray [Appl. Math. Comput. **132**, 643 (2002); Comput. Math. Appl. **60**, 1747 (2010)] and a numerical solution by employing a spectral scheme are presented and the results are depicted in a figure. The numerical results reveal that both solutions are in good agreement. © 2015 AIP Publishing LLC.

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I. INTRODUCTION

The idea of electron-acoustic mode had been conceived by Fried and Gould¹ during numerical solutions of the linear electrostatic Vlasov dispersion equation in an unmagnetized, homogeneous plasma. Besides the well-known Langmuir and ion-acoustic waves, they noticed the existence of a heavily damped acoustic-like solution of the dispersion equation. It was later shown that in the presence of two distinct groups (cold and hot) of electrons and immobile ions, one indeed obtains a weakly damped electron-acoustic mode (Watanabe and Taniuti²), the properties of which significantly differ from those of the Langmuir waves.

To study the properties of electron-acoustic solitary structures Dubouloz *et al.*³ considered a one-dimensional, unmagnetized collisionless plasma consisting of cold electrons, Maxwellian hot electrons, and stationary ions. However, in practice, the hot electrons may not follow a Maxwellian distribution due to the formation of phase space holes caused by the trapping of hot electrons in a wave potential. Accordingly, in most space plasma, the hot electrons follow the trapped/vortex-like distribution (Schamel^{4,5} and Abbasi *et al.*⁶). Therefore, in the present work, we shall consider a plasma model consisting of a cold electron fluid, hot electrons obeying a non-isothermal (trapped/vortex-like) distribution, and stationary ions.

The propagation of small-but-finite amplitude planar waves in a plasma with one-dimensional ion-acoustic model had been studied by several scientists (see, for instance, Washimi and Taniuti⁷) and in a plasma with one-dimensional electron-acoustic model by Schamel,^{4,5} Mamun and Shukla,⁸ Demiray⁹ by use of the classical reductive perturbation method (Taniuti¹⁰). Due to nonlinearity of the governing equations, in the long-wave limit, one obtains the KdV or modified KdV

equations for the lowest order term in the perturbation expansion, the solution of which may be described by solitons.

The propagation of small-but finite amplitude non-planar waves in various plasmas had been studied by several researchers in the current literature (see, for instance, Refs. 11–18). In all these works, the nonlinearities are of integer order, and then, the cylindrical KdV equation can be reduced to the conventional KdV equation through the use of some transform techniques. However, when the plasma is composed of a cold electron fluid, hot electrons obeying a non-isothermal (trapped/vortex-like) distributions and stationary ions, the nonlinearity of the cylindrical KdV equation is of fractional order and it cannot be reduced to conventional KdV equation.

In the present work, we consider the propagation of nonlinear electron-acoustic non-planar waves in a plasma composed of a cold electron fluid, hot electrons obeying a trapped/vortex-like distribution, and stationary ions. The basic equations of above described plasma, in the long-wave limit, are re-examined through the use of reductive perturbation method. The modified cylindrical KdV equation with fractional power nonlinearity is obtained as the evolution equation. Due to the fractional power nonlinearity such an evolution equation cannot be reduced the conventional KdV equation. Motivated with the approximate analytical method (in integral sense) developed by Demiray,^{19,20} we presented a progressive wave solution and a numerical solution by application of the spectral scheme to the evolution equation, and the results are illustrated in Figure 1. The results of both approaches reveal that they are in good agreement.

II. BASIC FIELD EQUATIONS

The dynamics of electron-acoustic waves is governed by the following equations:

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}) = 0, \quad (1)$$

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$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \alpha \nabla \phi = 0, \quad (2)$$

$$\nabla^2 \phi = \frac{1}{\alpha} n_c + n_h - \left(1 + \frac{1}{\alpha}\right), \quad (3)$$

where n_c is the normalized cold electron number density, n_h is the normalized hot electron number density, u_c is the cold electron fluid velocity, ϕ is the electrostatic potential, and the coefficient α is defined by $\alpha = n_{h0}/n_{c0}$, where n_{c0} and n_{h0} are the equilibrium values of the cold and hot electron number densities, respectively. The hot electron number density n_h (for $\beta < 0$) can be expressed by Schamel^{4,5}

$$n_h = I(\phi) + \frac{2}{\sqrt{-\pi\beta}} W_D(\sqrt{-\beta\phi}), \quad (4)$$

where

$$I(x) = [1 - \operatorname{erf}(\sqrt{x})] \exp(x),$$

$$W_D(x) = \exp(-x^2) \int_0^x \exp(y^2) dy, \quad (5)$$

where $\operatorname{erf}(x)$ is the error function. For $\phi \ll 1$, Equation (4) gives

$$n_h = 1 + \phi - \frac{4}{3\sqrt{\pi}} (1 - \beta) \phi^{3/2} + \frac{\phi^2}{2} - \frac{8}{15\sqrt{\pi}} (1 - \beta^2) \phi^{5/2} + \frac{\phi^3}{6} + \dots \quad (6)$$

In this work, we shall study the axially symmetric plasma of infinite length. As a result of this assumption, the field variables will be independent of θ and z coordinates in the cylindrical polar coordinates system. For this case, the field equations take the following form:

$$\frac{\partial n_c}{\partial t} + \frac{\partial}{\partial r} (n_c v) + \frac{1}{r} (n_c v) = 0, \quad (7)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} - \alpha \frac{\partial \phi}{\partial r} = 0, \quad (8)$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = -\frac{1}{\alpha} + \frac{n_c}{\alpha} + \phi - \frac{4}{3\sqrt{\pi}} (1 - \beta) \phi^{3/2} + \dots, \quad (9)$$

where $v_r = v$ is the radial velocity component.

For the asymptotic analysis of the field equations, we shall utilize the reductive perturbation method and introduce the following stretched coordinates:

$$\epsilon^{1/2}(r - t) = \xi, \quad \tau = \epsilon^{3/2} r. \quad (10)$$

Then, the following differential relations hold true:

$$\frac{\partial}{\partial t} = -\epsilon^{1/2} \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial r} = \epsilon^{1/2} \left(\frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial \tau} \right). \quad (11)$$

We further assume that the field quantities may be expressed as asymptotic series in ϵ as

$$v = \epsilon^2 (v^{(1)} + \epsilon v^{(2)} + \epsilon^2 v^{(3)} + \dots),$$

$$n_c = 1 + \epsilon^2 (n_c^{(1)} + \epsilon n_c^{(2)} + \epsilon^2 n_c^{(3)} + \dots),$$

$$\phi = \epsilon^2 (\phi^{(1)} + \epsilon \phi^{(2)} + \epsilon^2 \phi^{(3)} + \dots). \quad (12)$$

Introducing the expansions (11) and (12) into the field equations (7)–(9) and setting the coefficients of like powers of ϵ equal to zero, the following sets of differential equations are obtained:

$O(\epsilon^2)$ equations

$$-\frac{\partial n_c^{(1)}}{\partial \xi} + \frac{\partial v^{(1)}}{\partial \xi} = 0, \quad \frac{\partial v^{(1)}}{\partial \xi} + \alpha \frac{\partial \phi^{(1)}}{\partial \xi} = 0,$$

$$n_c^{(1)} = -\alpha \phi^{(1)}. \quad (13)$$

$O(\epsilon^3)$ equations

$$-\frac{\partial n_c^{(2)}}{\partial \xi} + \frac{\partial v^{(2)}}{\partial \xi} + \frac{v^{(1)}}{\tau} + \frac{\partial v^{(1)}}{\partial \tau} = 0,$$

$$\frac{\partial v^{(2)}}{\partial \xi} + \alpha \frac{\partial \phi^{(2)}}{\partial \xi} + \alpha \frac{\partial \phi^{(1)}}{\partial \tau} = 0,$$

$$n_c^{(2)} = \alpha \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - \alpha \phi^{(2)} + \frac{4\alpha}{3\sqrt{\pi}} (1 - \beta) (\phi^{(1)})^{3/2}. \quad (14)$$

A. Solution of the field equations

From the solution of the set (13) we have

$$v^{(1)} = n_c^{(1)} = -\alpha \phi_1, \quad (15)$$

where ϕ_1 is an unknown function whose governing equation will be obtained later. To obtain the solution for $O(\epsilon^3)$ equations, we introduce (16) into (14) to have

$$-\frac{\partial n_c^{(2)}}{\partial \xi} + \frac{\partial v^{(2)}}{\partial \xi} - \alpha \frac{\phi_1}{\tau} - \alpha \frac{\partial \phi_1}{\partial \tau} = 0,$$

$$\frac{\partial v^{(2)}}{\partial \xi} + \alpha \frac{\partial \phi_2}{\partial \xi} + \alpha \frac{\partial \phi_1}{\partial \tau} = 0,$$

$$n_c^{(2)} = \alpha \frac{\partial^2 \phi_1}{\partial \xi^2} - \alpha \phi_2 + \frac{4\alpha}{3\sqrt{\pi}} (1 - \beta) \phi_1^{3/2}, \quad (16)$$

where $\phi_2(\xi, \tau)$ is another unknown function whose governing equation will be obtained from higher order expansion. From the solution of these equations, one obtains

$$n_c^{(2)} = -\alpha \phi_2 + \alpha \left[\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{4}{3\sqrt{\pi}} (1 - \beta) \phi_1^{3/2} \right],$$

$$v^{(2)} = -\alpha \phi_2 + \frac{\alpha}{2} \left[\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{4}{3\sqrt{\pi}} (1 - \beta) \phi_1^{3/2} \right] + \frac{\alpha}{2\tau} \int \phi_1 d\xi, \quad (17)$$

and the following evolution equation:

$$\frac{\partial \phi_1}{\partial \tau} + \frac{\phi_1}{2\tau} + \frac{(1 - \beta)}{\sqrt{\pi}} \phi_1^{1/2} \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2} \frac{\partial^3 \phi_1}{\partial \xi^3} = 0. \quad (18)$$

This evolution equation is known as the modified cylindrical KdV equation.

B. Progressive wave solution

In this sub-section, we shall seek a progressive wave solution to the modified cylindrical KdV equation given in

(18). As is well-known the planar form of the modified KdV equation for such a plasma has the form^{5,9}

$$\frac{\partial u}{\partial \tau} + \frac{(1-\beta)}{\sqrt{\pi}} u^{1/2} \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^3 u}{\partial \xi^3} = 0. \quad (19)$$

This equation admits the solitary wave solution of the form

$$u = a \operatorname{sech}^4 \zeta, \quad \zeta = \kappa(\xi - c\tau), \quad (20)$$

$$\kappa^2 = \frac{(1-\beta)a^{1/2}}{15\sqrt{\pi}}, \quad c = \frac{8(1-\beta)a^{1/2}}{15\sqrt{\pi}},$$

where a is the constant wave amplitude.

Motivated with the solution given in (20), we shall propose a solution to Equation (18) of the following form:^{19,20}

$$\phi_1 = a(\tau) \operatorname{sech}^4 \zeta, \quad \zeta = \kappa(\tau)[\xi - c(\tau)], \quad (21)$$

where $a(\tau)$, $\kappa(\tau)$, and $c(\tau)$ are some unknown functions to be determined from the solution. Introducing (21) into (18) we obtain

$$\left[\left(\frac{a}{2\tau} + a' \right) - 4 \frac{\kappa'}{\kappa} a \zeta \tanh \zeta \right] \operatorname{sech}^4 \zeta + 4a\kappa \left\{ (c' - 8\kappa^2) \operatorname{sech}^4 \zeta + \left[15\kappa^2 - \frac{(1-\beta)a^{1/2}}{\sqrt{\pi}} \right] \operatorname{sech}^6 \zeta \right\} \tanh \zeta = 0, \quad (22)$$

where a prime denotes the differentiation of the corresponding quantity with respect to its argument. The solitary wave solution of the planar case corresponds to the expression in the last bracket; thus, we shall set it to zero and obtain the following equations:

$$\left[\left(\frac{a}{2\tau} + a' \right) - 4 \frac{\kappa'}{\kappa} a \zeta \tanh \zeta \right] \operatorname{sech}^4 \zeta = 0, \quad (23)$$

$$(c' - 8\kappa^2) \operatorname{sech}^4 \zeta + \left[15\kappa^2 - \frac{(1-\beta)a^{1/2}}{\sqrt{\pi}} \right] \operatorname{sech}^6 \zeta = 0. \quad (24)$$

From the solution of (24), we obtain the following relations:

$$\kappa^2 = \frac{(1-\beta)a^{1/2}}{15\sqrt{\pi}}, \quad c' = \frac{8(1-\beta)a^{1/2}}{15\sqrt{\pi}}. \quad (25)$$

These expressions are formally the same with those of Equation (18).

To obtain an additional equation for $a(\tau)$ we shall use (23), but this equation cannot be satisfied point by point. Following the procedure given in Refs. 19 and 20, we shall multiply the Equation (23) by $\operatorname{sech}^4 \zeta$ and integrate the result over ζ from $\zeta = -\infty$ to $\zeta = \infty$ to obtain

$$\left(\frac{a}{2\tau} + a' - \frac{\kappa'a}{2\kappa} \right) \langle \operatorname{sech}^8 \zeta \rangle = 0, \quad \langle \operatorname{sech}^8 \zeta \rangle = \int_{-\infty}^{\infty} \operatorname{sech}^8 \zeta d\zeta, \quad (26)$$

where $\operatorname{sech}^4 \zeta$ is a square integrable function and the integral is different from zero; thus, we have

$$\frac{a}{2\tau} + a' - \frac{\kappa'a}{2\kappa} = 0, \quad \text{or,} \quad \frac{a'}{a} - \frac{\kappa'}{2\kappa} + \frac{1}{2\tau} = 0. \quad (27)$$

Here, we note that the Equation (23) is satisfied in integral sense, but not point-wise. From the solution of (25) and (27) we obtain

$$a = a_0 \tau^{-4/7}, \quad \kappa^2 = \frac{(1-\beta)a_0^{1/2}}{15\sqrt{\pi}} \tau^{-2/7}, \quad c = \frac{56(1-\beta)a_0^{1/2}}{75\sqrt{\pi}} \tau^{5/7}, \quad (28)$$

where a_0 is a positive constant.

Thus, the final solution may be expressed by

$$\phi_1 = a_0 \tau^{-4/7} \operatorname{sech}^4 \zeta, \quad \zeta = \frac{(1-\beta)^{1/2} a_0^{1/4}}{(15\sqrt{\pi})^{1/2}} \tau^{-1/7} \left(\xi - \frac{56(1-\beta)a_0^{1/2}}{75\sqrt{\pi}} \tau^{5/7} \right). \quad (29)$$

C. Numerical solution

In order to check the validity of the derived analytical solution of the modified cylindrical KdV equation given in (19), we implement a numerical scheme. The implemented scheme is a spectral scheme with a 4th order Runge-Kutta time integrator as discussed in Bayındır²¹ and Karjadi *et al.*²² For the time integration by a 4th order Runge-Kutta scheme, we first rewrite the Equation (18) as

$$\frac{\partial \phi_1}{\partial \tau} = -\frac{\phi_1}{2\tau} - \frac{(1-\beta)}{\sqrt{\pi}} \phi_1^{1/2} \frac{\partial \phi_1}{\partial \xi} - \frac{1}{2} \frac{\partial^3 \phi_1}{\partial \xi^3} = g(\phi_1, \tau), \quad (30)$$

where g is the function which represents the right-hand-side of the Equation (18). Starting with the numerical values of the functions at an arbitrary time step n , the numerical values for the next time step $n+1$ can be written as

$$\phi_1^{n+1} = \phi_1^n + \frac{\Delta \tau}{6} (s_1 + 2s_2 + 2s_3 + s_4) \quad (31)$$

and

$$\tau^{n+1} = \tau^n + \Delta \tau, \quad (32)$$

where $\Delta \tau$ is the time step. The value of $\Delta \tau = 0.01$ is selected in this study and scheme is stable with this value. s_1, s_2, s_3, s_4 are the slopes which represent the time derivatives in a time step and they are given by

$$s_1 = g(\tau^n, \phi_1^n), \quad (33)$$

$$s_2 = g(\tau^n + \Delta \tau/2, \phi_1^n + s_1 \Delta \tau/2), \quad (34)$$

$$s_3 = g(\tau^n + \Delta \tau/2, \phi_1^n + s_2 \Delta \tau/2), \quad (35)$$

$$s_4 = g(\tau^n + \Delta \tau, \phi_1^n + s_3 \Delta \tau). \quad (36)$$

Starting from the initial conditions, time stepping is performed by the 4th order Runge-Kutta scheme summarized above.

The spatial derivatives are handled by a spectral method in the wavenumber space. Considering a periodic domain,

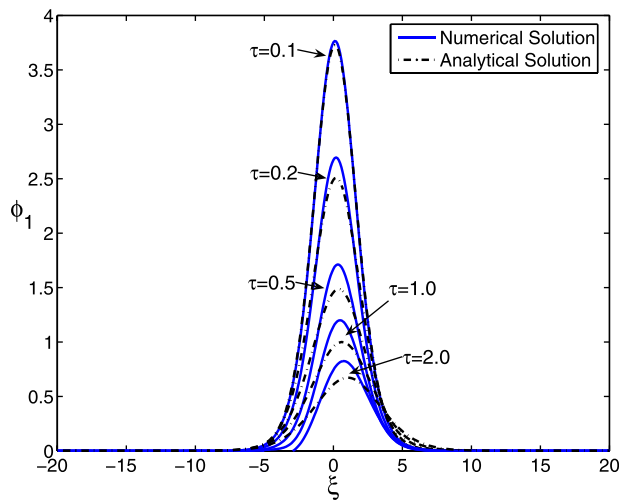


FIG. 1. The variation of wave profiles: comparison of the numerical and analytical solutions of the modified cylindrical KdV equation for different time values.

Fourier representations of the unknown function ϕ_1 is considered so that the spectral derivatives can be calculated by

$$\frac{\partial \phi_1}{\partial \xi} = F^{-1}[ikF[\phi_1]]$$

and

$$\frac{\partial^3 \phi_1}{\partial \xi^3} = F^{-1}[-ik^3 F[\phi_1]], \quad (37)$$

where F and F^{-1} denote the Fourier and the inverse Fourier transform operations, respectively. k is the wavenumber vector which includes exact N multiples of the fundamental wavenumber $k_o = 2\pi/L$. The number of spectral components is selected as $N = 1024$ in order to make use of fast-Fourier transform algorithms efficiently. Domain length is selected as $L = 400$. All nonlinear products are calculated in the physical space by simple multiplication. A more detailed discussion of the spectral methods can be seen in Canuto *et al.*²³

III. RESULTS AND DISCUSSION

For the illustration of the analytical and numerical results, we choose $a_0 = 1$, $\beta = -0.5$ (see Schamel^{4,5}). The function ϕ_1 is calculated for time values of $\tau = 0.1, 0.2, 0.5, 1.0, 2.0$ as a function of ξ and both the analytical and numerical solutions are depicted on the Figure 1 below. As can be seen in the figure, the analytical solutions are in good agreement with the numerical solutions. Additionally, the wave amplitude decays very fast as one goes away from the center line of the cylinder. Same decaying behavior can be observed as the time progresses. For bigger values of τ the solution decays to zero progressively.

This is expected because of the cylindrical $-\partial \phi_1 / \partial \tau$ component in Equation (30). Both the numerical and the analytical results confirm that proposed solitary wave solution is a valid solution therefore we conclude that analytical solution and the procedure for deriving the analytical solution is correct and can be used as integral-wise correct solitary wave solutions of the modified cylindrical KdV equation.

IV. CONCLUSION

In this study, we considered the propagation of nonlinear non-planar electron-acoustic waves in a plasma which consists of a cold electron fluid, hot electrons obeying a trapped/vortex-like distribution, and stationary ions. The basic governing equations of the plasma described above, in the long-wave limit, are re-examined by utilizing reductive perturbation method. The modified cylindrical KdV equation with fractional power nonlinearity, which cannot be reduced to the conventional KdV equation, is obtained as the evolution equation. The integral-wise correct analytical progressive solitary wave solution of the modified cylindrical KdV equation is derived. Also, a spectral numerical scheme is implemented for the numerical solution of the modified cylindrical KdV equation and the analytical results are compared with the numerical results. Comparisons revealed that both solutions are in good agreement, thus derived analytical solution can be used as the integral-wise correct solitary wave solutions of the modified cylindrical KdV equation.

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